

III

ACHILLES AND THE TORTOISE

I SHALL now discuss a dilemma which I imagine is familiar to everybody. It is quite certain that a fast runner following a slow runner will overtake him in the end. We can calculate by simple arithmetic after what distance and after what time the chase will be over, given only the initial distance and the speeds of the two runners. The chase will be over in the time it would take to cover the initial interval at the speed of the fast runner minus the speed of the slow runner. The distance covered by the pursuer by the end of the chase is calculable from his actual speed over the ground and the time for which he runs. Nothing could be more decisively settled. Yet there is a very different answer which also seems to follow with equal cogency from the same data. Achilles is in pursuit of the tortoise and before he catches him he has to reach the tortoise's starting-line, by which time the tortoise has advanced a little way ahead of this line. So Achilles has now to make up this new, reduced lead and does so; but by the time he has done this, the tortoise has once again got a little bit further ahead. Ahead of each lead that Achilles makes up, there always remains a further, though always diminished lead for him still to make up. There is no number of such leads at the end of which no lead remains to be made up. So Achilles never catches the tortoise. He whittles down the distance, but never whittles it down to nothing. Notice that at each stage the tortoise's lead is a finite one. If Achilles has whittled off ten or a thousand such ever dwindling leads one after another, the lead still to be made up is of finite length. We cannot say that after such and such a number of stages, the tortoise's lead will have shrunk to the dimensions of an Euclidean point. If Achilles takes any time at all to make up a lead, he gives time for the tortoise to get some way past the terminus of that lead. The same result follows if we consider intervals of time instead of distances in space. At the end of the period taken to make up one lead, there remains another diminished period in which Achilles has to make up the next lead. There is no finite number of such ever-

diminishing overtaking-periods, such that we can say that after 100 or 1000 of them, no further period of pursuing remains.

This is one of the justly famous paradoxes of Zeno. In many ways it deserves to rank as the paradigm of a philosophical puzzle. It clearly is a philosophical puzzle and not an arithmetical problem. No solution is to be looked for by going over, with greater care, the calculations by which it is established that Achilles will catch the tortoise in, say, exactly six minutes. But nor is a solution to be found by reconsidering the argument proving that lead 1 plus lead 2, plus lead 3, etc., never add up to the total distance to be covered by Achilles in order to catch the tortoise. There is no number, such as a million, such that after a million of these dwindling leads have been made up, no lead remains to be made up.

I shall try to expose just where Zeno's argument seems to prove one thing, namely that the chase cannot end, but really proves, perfectly validly, a different and undisturbing conclusion; and also to show why the difference between this real and that apparent conclusion is in an interesting way a queerly elusive one. It is the elusiveness of this difference which makes it so excellent a specimen of a logical dilemma.

In offering a solution of this paradox, I expect to meet the fate of so many who have tried before, namely demonstrable failure. But for my general purpose this will not matter. I shall have exhibited that the argument is a tricky one, and brought out for consideration some of the factors which make it tricky. Even if I fail, I may with luck have betrayed, without knowing it, some factor which has succeeded in tricking me.

First, let us notice some seemingly trivial points which the two conflicting treatments of the race have in common, or seem to have in common. To make the question definite, let us suppose that Achilles runs at eleven miles an hour, while the tortoise crawls at one mile an hour, and that the tortoise has a start of one mile. According to the natural treatment, the race will be over in the time that it would have taken Achilles to reach the tortoise if the tortoise had not budged at all, and Achilles had run at ten miles an hour instead of eleven miles an hour; i.e. the race will be over in one-tenth of an hour, or six minutes. As Achilles runs for six minutes at eleven miles an hour, the

distance he will have covered is eleven tenths of a mile, i.e. a mile and a tenth of a mile. This calculation is done in terms of miles and fractions of miles, and in hours and fractions of hours, namely minutes. But obviously it would have made no difference had we instead worked the distances out in yards and inches, or metres and centimetres, or if we had worked out the times in seconds or in fractions of a year.

The same thing is true of Zeno's treatment of the race. The racers start a mile apart (or 1760 yards apart or 5280 feet or the corresponding number of metres or centimetres). While Achilles runs the initial mile, the tortoise crawls his fraction of a mile, namely one eleventh of a mile; while Achilles is covering this next fraction of a mile, the tortoise is crawling his next fraction of that fraction of a mile, and so on. For each successive lead that Achilles has made up, the tortoise has established a new lead of his regular fraction of the length of the one before.

That is to say, in both treatments our calculations are calculations of distances and parts of those distances, e.g. miles and elevenths of a mile, or furlongs and elevenths of a furlong; or they are calculations of stretches of time and parts of those stretches, e.g. hours and fractions of hours or minutes and fractions of minutes.

According to the natural treatment, the race is over in six minutes. Its duration consisted of the first minute, plus the second minute, plus the third . . . up to six. These parts of that duration duly add up to the whole. If instead we partition the duration of the race into seconds, it would come to 360 seconds, and these 360 parts duly add up to the whole six minutes. Similarly the total distance run by Achilles is a mile and a tenth or, if you prefer, 1936 yards; and the tenths of a mile covered (or the yards) duly add up to the total.

Here I am simply reminding you of the platitude that a whole is the sum of its parts, or that $\frac{12}{12} = 1$, or $\frac{1936}{1936} = 1$, or, generally, whatever number x stands for, $\frac{x}{x} = 1$.

But to our consternation according to Zeno's treatment of the race, this platitude seems to break down. Here again we have stretches of space and sub-stretches of it, or stretches of

time and sub-stretches of it. Yet here the slices that we have cut off refuse to add up to the whole distance or the whole duration. The first lead that Achilles makes up, plus the second, plus the third . . . never add up to the distance required for him to have caught the tortoise. Wholes surely are sums of their parts, yet here are parts of a whole which, however numerous, never amount to that whole. Or a whole *is* all of its parts taken together; yet here we have as many parts as we like, but such that we can at no stage say that we have now taken together all of them. For there is at every stage a part left outstanding.

Let us consider, for a moment, the slices into which a cake may be cut. Cut the cake into six or sixty slices, and these six or sixty slices, taken together, constitute the entire cake. The cake is its six sixths or its sixty sixtieths. But now suppose that the mother of a family chooses instead to circulate an uncut cake round the table, instructing the children that each is to cut off a bit and only a bit of what is on the cake-plate; i.e. that no child is to take the whole of what he finds on the plate. Then, obviously, so long as her instructions are observed, however far and often the cake circulates, there is always a bit of cake left. If they obey her orders always to leave a bit, then they always leave a bit. Or to put it the other way round, if they obey her orders never to take the whole of the last fragment, a fragment always remains untaken. What they have taken off the cake-plate never constitutes the whole cake. Certainly what they take at each helping is a part—a steadily diminishing part—of the whole cake. But the cake is, at any selected stage, not merely the sum of these consumed parts. None the less it really is the sum of these consumed parts *plus the unconsumed part*. This addition sum works out correctly at each stage at which the cake-plate is passed on. At that moment the pieces already taken plus the fragment still untaken do constitute the whole cake. Similarly at the next stage, and the next. But at no stage is the unconsumed residue not a proper part of the cake; so at no stage do the parts consumed amount to all of the parts of the cake. This is simply the platitude that a whole is more than the sum of *all of its parts but one*, however small that one may be. The mother's second method of cake-partition ensured that there should be at every stage such a part left on the cake-plate,

though one of smaller and smaller dimensions with each stage of the division.

She could make her instructions more precise. She now passes the plate round the children in order of decreasing seniority, and in order that bigger children shall have the bigger portions, she instructs the children always to take not just a bit but exactly half and so to leave just half of what is on the plate. The first child begins with a half cake, and leaves a half, the second gets a quarter, and leaves a quarter, the third gets an eighth, and leaves an eighth, and so on. The plate never stops circulating. After each cut there remains a morsel to be bisected by the next child. Obviously the children's patience or their eyesight will give out before the cake gives out. For the cake cannot give out on this principle of division.

Notice again, that while the slices taken at no stage amount to the whole cake, yet at each stage the slices so far taken *plus the morsel still untaken* certainly do amount to the whole cake. These slices taken plus the morsel remaining can be counted, so that at each stage we can speak in the ordinary way of all the parts of the cake, namely, say, all the 99 slices already taken plus the one morsel now outstanding, i.e. 100 bits in all. At the next stage the scope of the 'all' will be different. It will now be all the 100 slices taken plus the crumb now outstanding, i.e. 101 bits in all. There is another point to be borne in mind for future use. The size of each slice, if the bisection is exact, is a measurable and calculable fraction of the size of the original whole cake; the first slice to be eaten was a half-cake, the second was a quarter-cake, the third was an eighth of the cake, and so on. The sizes of the slices are fixed in terms of the size of the cake. The partition-method employed was from the start a method of operating upon the cake as a whole. So if, say, the second child, playing the Zeno, were to say 'What we consume never amounts to the whole cake; so I believe that there never was a whole cake of finite size to consume', he could be refuted by being asked what his own first slice was one-quarter of. There must have been the whole cake, for him to get a quarter of it; and a finite one, for his quarter of it was finite. Or he could be asked what it is, according to him, that the parts consumed never amount to.

I now want to satisfy you that the race between Achilles and

the tortoise exemplifies just what is exemplified by the mother's division of the cake by the second method.

In order both to simplify the story and to bring it into parallel with the second method of dividing the cake, let us now say that Achilles saunters at two miles an hour, the tortoise crawls at one mile an hour, and has a start of one mile. Since the difference between their speeds is one mile an hour, Achilles will catch the tortoise in one hour, by which time he will have covered two miles of the race-track. Now we spectators of the race might, after the event, go back over this two-mile course of his and plant a flag in the ground at the end of each of the eight quarter-miles, or each of the sixteen furlongs that Achilles had run. Our last flag would then be planted where the race ended. But now suppose that, when the race is over, we go back over these two miles of the track covered by Achilles, and choose instead to stick one flag into the ground where Achilles started, a second at the half-way point of his total course, a third at the half-way point of the second half of his course, a fourth at the half-way point of the outstanding quarter of his course, and so on. Clearly for every flag we plant, there is always another flag to put in half-way between it and the place where Achilles caught the tortoise. (In fact, of course, we shall soon reach a point where our flags are too bulky for us to continue the operation.) We shall never be able to plant a flag just at the place where the race ended, since our principle of flag-planting was that each flag was to be planted half-way between the last flag planted and the place where the race ended. In effect our instructions were to plant each flag ahead of the last one but also behind the terminus of the race. If we obey these instructions, it follows that we never plant a flag which is not behind the terminus, and so that we never plant the last flag. At no stage does the distance between Achilles' start-line and the last flag to be planted amount to the whole distance run by Achilles. But conversely, at each stage the total distance run by Achilles does consist of the sum of all the distances between the flags *plus the distance between the last flag planted and the point where the race ended*. Achilles' whole course is not the sum of all of its parts but one; it is the sum of all of those flagged parts plus the outstanding unflagged one. The number of these stretches alters and the

length of both the last stretch to be flagged and the remainder-stretch alters with each new flag that is planted. At one stage, $\frac{15}{16}$ of his course has been flagged and $\frac{1}{16}$ of his course is still ahead of the last flag planted, and $\frac{15}{16} + \frac{1}{16}$ duly = 1. At the next stage $\frac{3}{2}$ of his course has been flagged and $\frac{1}{2}$ of his course is still ahead of the last flag planted; but again $\frac{3}{2} + \frac{1}{2}$ duly = 1.

No great mystery seems to confront us here. If we obey the instruction always to leave room for one more flag, we always leave room for one more flag. Nor can the fact that no flag is the last flag persuade us that Achilles' course was endless, since we knowingly began our flag-planting operations with the datum that his was a two-mile course, the start-line and the terminus of which we knew. The places where we planted our flags were fixed in terms of just this two-mile course, namely one flag at its midpoint, the next at the end of its third half-mile, the next at the end of its fourteenth furlong and so on. We were, all the time, planting flags to mark out determinate portions of the precise two-mile course that Achilles ran. We could, if we had chosen, have worked backwards on the same principle from the terminus of the race; and then we should never get a flag planted on his start-line. Yet this would not persuade us that a race had a finish, but no beginning.

What the distances flagged fail at each stage to amount to is the two-mile distance that he had run by the time he caught the tortoise, just because this distance is, according to the instructions, the sum of those flagged distances plus whatever unflagged distance remains outstanding.

It is easy now to see that the flags planted according to these instructions do in fact mark precisely the termini of those successive leads established by the tortoise on which Zeno made us concentrate. From Achilles' start-line to the tortoise's start-line was just the mile between the first flag that we planted and the second. Where the tortoise was, when Achilles reached this half-way point of his total course, is the place where we planted our flag for the third quarter of Achilles' total chase, and so on. What we measure off after the event with a surveyor's chain and, later on, a micrometer, Achilles might in principle, though not in practice, have measured off by running steadily at twice the tortoise's speed and by marking, in his mind, the termini of

the tortoise's successive leads. If informed that he was going at twice the tortoise's speed, then Achilles himself could have known, while running, that the terminus of the first lead was the midpoint of his pursuit, that the terminus of the tortoise's second lead marked the third quarter of his pursuit, that the next marked the seventh eighth of his pursuit, and so on. Given their actual speeds, he would have known that he would catch the tortoise at the second milestone, and so that the successive leads were determinate portions of what was going to be his two-mile chase. But we are induced to imagine that Achilles was without these data by the fact that in ordinary races the runners do not know just how fast they or their opponents are running; they do not know that their opponents are not accelerating or decelerating or just about to stop or even to start coming backwards. But had he known what we are allowed to know, that his and his opponent's speeds were constant, and that his speed was twice his opponent's, then he himself could have used his own progress from lead-terminus to lead-terminus as, so to speak, a moving surveyor's chain; and he could have recognized the termini of the successive leads that he had to make up as doing just what our flags do, namely as marking off determinate slices of his total course from start-line to the terminus of his pursuit. The series of these diminishing leads would then have felt to him not like an endless sequence of postponements of victory but like, what of course they were, measured stages towards his calculable victory. Just this is part of Zeno's trick. Zeno professed to be trying to build up Achilles' total course out of this series of leads made up, where we have been dividing up Achilles' total two-mile course, taken as our datum, by a flag-planting procedure each stage of which was, by rule, non-ultimate. We chose to apply a special partition-procedure to a known and determinate stretch of a race-track, namely the two miles of it that Achilles ran; we cannot, therefore, be browbeaten by the interminableness of the task of flag-planting into doubting whether Achilles' pursuit had a terminus. Zeno, ingeniously, started at the other end. By talking in terms of distances still to be covered by Achilles, he got the endlessness of this series of leads to browbeat Achilles and us into doubting whether he could catch the tortoise at all. Yet the termini of the

successive leads that Achilles has to make up according to Zeno's account come exactly where we planted our flags to mark out our regularly diminishing but determinate slices of Achilles' precise two-mile course to victory. In other words, Zeno has, ingeniously, got us to look at our flag-system back to front, rather as if the mother told her children that she had made her cake that morning by assembling the eldest child's half-cake, the second child's quarter-cake, the third child's eighth, and so on—a story which they would quickly see through, not only because the morsel still on the cake-plate is going to be left out of her inventory, but also because in her very mention of the eldest child's half-cake and the second child's quarter-cake, and so on, she had already been referring to the whole cake, as that whole of which their determinate portions had been those determinate portions. Similarly Zeno, in his mentions of the successive leads to be made up by Achilles, is, though surreptitiously and only by implication, referring to the total two-mile course run by Achilles in overtaking the tortoise; or in other words, his argument itself rests on the unadvertised premiss that Achilles does catch the tortoise in, say, precisely two miles and in precisely one hour. For he has told us that Achilles is overtaking the tortoise at one mile an hour, and that the initial lead was one mile. As I said, the reason why at first sight this does not seem to be the case is that we are induced to look at the race through Achilles' own eyes. He can see, we suppose, where the tortoise's start-line is all the way from his own start-line. As he reaches the tortoise's start-line, he can see the terminus of the new lead that the tortoise has now established, and so on. But he cannot at any stage see a tape to be broken by the winner of the race, since what in this race corresponds to reaching the tape in an ordinary side-by-side race, is his catching up with the tortoise, and where on the race-track this will occur is not a visible feature of the track. So, unless he knows what we have been told, he cannot be thinking of the successive leads as calculable fractions of his eventual total course, in the way in which the mother, if she has kept count, can calculate the weights of the successive portions cut off the cake as specified fractions of the weight of the original cake. She weighed the cake before tea; Achilles did not measure his run before he made it, and we are

induced to assume that he could not know its length while running it. The mother, knowing the weight of the cake and the scrupulousness of the bisection of the slices taken, can, just by keeping count of the cuts, also keep tally, stage by stage, of the weights of the slices removed and thence of the weight of the remainder of the cake on the cake-plate. But Achilles who does not, we naturally assume, know precisely his own speed or that of the tortoise, even if he does know the exact length of the tortoise's start, cannot work out exactly when he has covered the first half, the first three-quarters, the first seven-eighths, etc., of what will have been his total course to victory. If we gave him our flags to drop as he reached these points, he would not know just where to drop them. Yet if he does drop them just at the terminus of each of the leads which Zeno describes him as making up, one after the other, he will in fact unwittingly have dropped them just where we, after the event, would have deliberately planted them. Our chosen principle of flag-planting is just the obverse of the facts, which presumably Achilles does not know, that his speed is double that of the tortoise, and so that the tortoise's one-mile start constitutes just half of what is to be Achilles' total course. The lengths of the successive leads that Achilles has to make up are necessarily proportional to the difference between the speeds of the two competitors. Achilles himself is more nearly in the position of the mother, if she had instructed her children merely to take a bit and leave a bit of whatever remains on the plate, without prescribing any scale for these bits. She cannot now calculate the actual weight of the portions consumed or of the portion still unconsumed. But she still knows that at each stage the combined weights of the consumed portions and the unconsumed portion, whatever these may be, add up to the weight of the original cake.

Similarly the whole of the course that Achilles will have run is indeed the sum of as many parts as we or he may care to slice off it plus the part that we or he have left on it. The fact that these parts are of diminishing length, on this principle of partition, is of no more interest than the fact that the parts were all of the same length on our first principle of partition. As a cake is not five of the six slices into which it has been cut, but those five plus the remaining sixth slice, so Achilles' course is not the sum of

the half, plus the quarter, plus the eighth of it, etc., that we have at this or that stage chosen to put on one side, but it is the sum of these plus the remainder. Nor is this remainder one of mysterious or elusive dimensions. It is of exactly the same dimensions as the last fraction that we sliced off before we chose to stop slicing.

Certainly, if we choose to conduct our slicing according to the principle that a remainder shall always be left, a remainder is always left. That this division can go on *ad infinitum* is an alarming phrase, but it means no more than that after each cut, a remainder is left to divide by a subsequent cut. But the consoling truth remains that whether we stop after two cuts or after two hundred, the whole off which we were cutting is the sum of what we have cut off it, plus what we have left.

To put a central point very crudely, we have to distinguish the question 'How many portions have you cut *off* the object?' from the question 'How many portions have you cut it *into*?' The answer to the second question is higher by one than the answer to the first. The platitude 'a whole is the sum of its parts' means that a whole is the sum of the portions you cut it into; it does not mean, what is false, that it is the sum of the portions you have cut off it, if this phrase implies that something remains. Zeno gets us and Achilles to think of each of the successive leads that are to be made up as portions which ought somehow to add up, but cannot add up, to the total course he has to run. He thus averts our attention from the fact that these successive leads were, in effect, selected by Achilles for being only slices cut off the distance he has yet to run, i.e. for making up that total distance *minus* something. His principle of selection presupposes that there is the total distance which he has got to run—else there would be nothing for him to select as an intermediate slice of that distance. Suppose that as Achilles reaches the first milestone he sees the tortoise at the next half-mile post. According to Zeno, he argues despondently 'I have got to reach that half-mile post first and still run on a bit further in order to catch the tortoise'. But his argument assumes that he knows that the tortoise is not going to stop crawling at the half-mile post. If he does stop there, he will be caught there. Achilles is then supposed not only to know that the tortoise is now at that half-mile

post but also to assume that he is going past it, i.e. to assume that the half-mile post marks only some fraction of the distance to the terminus of the race. That there is a definite distance to that terminus is presupposed by his assumption that the half-mile post is only a part of that distance, i.e. that a lead to be made up is a stage towards the finish of the race, and therefore not the whole of the distance to that point.

Of course, if it is an ordinary race, Achilles may not catch the tortoise at all. He will not do so if he himself so slows down or the tortoise so accelerates that there is now no difference between their speeds, or else a difference in the tortoise's favour. But this is only to say that Achilles cannot overtake the tortoise without going faster than the tortoise—a thing which we and he never doubted. If the race does take this unfavourable turn, then the next half-mile post will indeed not mark a part of Achilles' total course to victory, since there is going to be no victory. It does mark off a part of his total course to victory, if and only if Achilles is in fact overtaking and going on overtaking the tortoise—a condition which was granted to us by Zeno, though perhaps not imparted to Achilles. So we assume that Achilles cannot know that the next lead to be made up is a definite fraction of what will be his total course to victory, since he does not know that he will win or that his speed is to be constant at twice that of the tortoise. But we have, by implication, been told that he will win, so we know that this lead, and the next and the next, are definite fractions of his total course to victory. But we were induced to put this knowledge into cold storage by being led to look at the race through Achilles' eyes. We were trying to envisage our surveyor's task through the haze of a runner's doubts, ignorances and despondencies. So we thought of his course as composed of an échelon of diminishing, intermediate stages, each of which, because intermediate, was therefore non-ultimate. We forgot, what we knew all the time, what these stages were intermediate between, namely between Achilles' start-line and the place where he caught the tortoise. We forgot that what is cut off the cake is not what the cake is cut into, and that as what had at each stage been cut off it was measured or calculable, so what, at that stage, the cake had been cut into was measurable or calculable.

Now let us draw some general lessons from this dilemma.

First of all, though it is presented in the dramatic form of a foot-race under Greek skies between two rather engaging characters, its argument is of quite general application. A race involves the covering of a distance in a time. Part of our confusion was due to our wondering whether we ought to be concentrating on furlongs or on minutes. But the argument applies where there is no question of the passage of time, as, for example, in the case of the progressive bisection of a cake. It applies, too, where there is no question of stretches of space, as, for example, in the case of an initially cool thermometer overtaking the rising temperature of the contents of a saucepan, or a clever junior overtaking the scholarship-level of a senior boy whose scholarship is improving too, though less rapidly.

Next, in this particular issue we are trying to find out after what stretch of time and after what distance Achilles overtakes the tortoise, or else, if scared by Zeno, we are trying to find out if there is any stretch of time or any distance at the end of which Achilles has overtaken him. In both cases we are thinking about or intellectually operating upon slices of a day and slices of a chase. In other applications we might be thinking about (or operating upon) slices of cake, or degrees of temperature.

But in an important way we are, in all applications, thinking *in terms of* or operating *with* the same overarching notions of *part, whole, fraction, total, plus, minus* and *multiplied by*. It is because we have already learned to execute some abstract manoeuvres with these notions, i.e. sums in simple arithmetic, that we are capable of calculating when a man will catch a tortoise and capable, too, of being embarrassed by an argument which seems to prove that he will never do so. A boy who has run and witnessed many races, but cannot yet grasp the abstract platitude that a whole is the sum of its parts, cannot yet work out how many quarter-miles there are in a two-mile course, nor can he grasp the other abstract platitude that the portions cut off something at no stage amount to the whole of that thing.

But now consider the boy who has reached the stage of dealing clear-headedly in simple, abstract arithmetic, not only

with fractions, and their addition and subtraction, but also with the multiplication of fractions. He realizes well enough, in the abstract, that not merely does $\frac{2}{3} \times \frac{2}{3}$ come to something less than 1, but even fractions like $\frac{9}{10} \times \frac{9}{10}$ or $\frac{999}{1000} \times \frac{999}{1000}$ come to something less than 1, and less even than either of the fractions themselves. Yet when the family cake is cut, not according to the usual principle of dividing it into six or ten more or less equal slices, but according to the unusual principle of so dividing it, that each cut divides the remainder in a given ratio, he may still get the feeling that the cake has been transformed into a magic cake, a cake which allows itself to be cut at and cut at for ever. It now seems to be an inexhaustible cake, and yet inexhaustible in a disappointing way, since the family gets no more cake, indeed somewhat less cake, than it did when it was cut in the usual way. Though there is always more cake to come, yet the cake has visibly not, like the Hydra, repaired its losses. That is to say, though he knows how to apply to such things as cakes, or two-mile stretches of a race-track, the simple, abstract notions of fractions and sums of fractions, he is not yet clear about the application to cakes or race-tracks of the more complex abstract notion of the products of fractions. He cannot clearly distinguish between the inexhaustibility of a magic cake or a magic race-track which repairs its losses, and the inexhaustibility of the series of a fraction of an ordinary cake or race-track plus that fraction of the remainder, plus that fraction of the remainder. . . . Confusedly, he attributes to the cake or race-track a difference from ordinary cakes and race-tracks, which is really a difference between one division procedure and another division procedure. He ascribes a queer endlessness to Achilles' pursuit of the tortoise, where he should have ascribed an uninteresting non-finality to each of the stages of a certain, special way of subdividing two miles.

He is behaving somewhat like the boy who, having learned one card game, namely 'Snap', when he comes to a new card-game, like Whist, cannot for a while help assimilating what he has to do with his cards now to the things he has long since learned to do with those same cards in 'Snap'. He is put out at finding that play which works in 'Snap', does not work in Whist, and vice versa. Yet, in a way, he *has* learned the rules of

Whist—he has learned them well enough for some purposes, but not well enough to be safe from relapsing now and again into ‘Snap’ play and ‘Snap’ thinking. After all, the cards he is playing with now are the same old cards.

This point brings us back to a suggestion that I made in the previous chapter, but left for later expansion. The collision between the natural view that Achilles catches the tortoise after a pursuit of measurable and calculable length and the queer view that he never catches him at all, does not occur while we are thinking at ground-floor level of such things as Achilles’ paces, the dusty furlongs of the track and the tortoise’s inferior speed. It occurs when we reach the first-floor level of thinking, on which we try to work out if and when Achilles will catch the tortoise by procedures of calculation which are of quite general application. The pony is docile enough in its home paddock. It is when we try to drive him in some standardized conceptual harness that his habits and our intentions conflict, even though we have got quite used to the pony’s behaviour in the paddock and, also, but separately, quite used in the harness-room to the construction and assemblage of the harness. Handling this conceptual pony in this conceptual harness involves us in troubles, for which we cannot fix the blame on either the pony or the harness. These excellent reins get under that excellent pony’s hooves. How is what we know quite well about the stages of an athlete’s victorious pursuit to be married with what we also know quite well about the results of adding together a fraction of a whole, that fraction of the remainder, that fraction of the next remainder, and so on?

For example, Zeno’s argument seems to prove that Achilles never catches the tortoise—never, in the sense that years, centuries, millennia after the start of the race Achilles will still be in hopeless pursuit; that the race is an eternal race, like the pursuit by a donkey of the carrot suspended in front of his nose. But this sense of ‘never’, in which all eternity is occupied in vain pursuit, is quite different from the sense of ‘never’, in which we say, when talking arithmetic, that the sum of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc., never amounts to unity. To say this is simply to utter the general proposition that any particular remainder-bisection leaves a remainder to bisect. The only connexion that this ‘never’ has

with the 'never' of all eternity is that if a silly computer were to attempt to continue bisecting remainders until he had found one which was halved but had no second half, his attempt would then go on to all eternity. Such a computer would indeed resemble the donkey pursuing a carrot which is suspended in front of his nose. But the arithmetical proposition itself says nothing about silly or sensible computers. It itself is not disheartening prophecy, for it is not a prophecy at all; it is just a general truth about a fraction.

A similar ambiguity belongs to the word 'all'. When a cake is divided in the ordinary way into six or sixty portions, we can speak of all these portions, and enumerate them. There are six or sixty of them in all. We have a countable total and it amounts to the whole cake. When the cake is divided according to the less usual principle, that each bit taken shall be only a fraction of what remained after the previous cut, then again we can use the word 'all' or 'total' in this same manner. We can talk about and enumerate the bits already taken at stage 3, or the bits already taken at stage 7 and so on. Here the bits already removed at this or that specified stage do not amount to the whole cake. At this or that given stage, what amount to the whole cake is the, still countable, total of the bits removed plus the one bit still on the plate. But for certain purposes we want to stand back from this or that specified stage of the division-process, and to talk about the procession of these stages. For example we want to say, quite generally, that all the cuts leave residues to be cut. Now here the 'all' is not a countable total—and it is not an uncountable total either. For it is not a total. What it expresses can be expressed just as well by 'any', namely 'any cut leaves a residue to be cut'.

That is, in the first use of 'all' we could, in principle, fill out with 'all six...', or 'all sixty...'. In this second use of 'all' we could not fill out with 'all (so and so many)...'. Not because there are *too* many, but because 'any...' carries, *ex officio*, the notion of 'no matter which...', and this is not a totality-notion of any sort, familiar or queer.

Unfortunately for us, we have here had to use both notions together, both that of 'all (so and so many)...' and that of 'any (no matter which)...'. For we have to say that at any

stage (no matter which), all the x bits then removed amount to something less than the whole cake; or that at any stage, no matter which, the total of the x bits taken plus the one bit untaken does amount to the whole cake.

We talk about a race in one tone of voice, we talk arithmetic in another tone of voice; but in talking the arithmetic of a race we have to mix our tones of voice, and in doing this we may easily feel—and even speak as if—we were talking out of different sides of our mouths at the same time.

We decide factual questions about the length and duration of a race by one procedure, namely measurement; we decide arithmetical questions by another procedure, namely calculation. But then, given some facts about the race established by measurement, we can decide other questions about that race by calculations applied to these measurements. The two procedures of settling the different sorts of questions intertwine, somehow, into a procedure for establishing by calculation concrete, measurable facts about this particular race. We have the pony in the harness that was meant for any such pony, yet we can mismanage the previously quite manageable pony in its previously quite manageable harness. Two separate skills do not, in the beginning, intertwine into one conjoint skill.

Looking back, now, at the fatalist imbroglio which we expressed in the slogan 'Whatever is, always was to be', we can see without difficulty that here too our trouble was a sort of pony-harness trouble. The platitude that whatever happens would have fulfilled any prior guess to the effect that it would happen was a logician's platitude. It gave us no news about what happens, but it told us a truism about what it is for a statement in the future tense to come true. On the other hand, the platitudes that many things that happen are our fault and that there are some catastrophes which can and others which cannot be averted, these are not logicians' truisms, but truisms about the world and human beings. Very crudely, they are nursemaids' truisms. In attempting to harness the nursemaid's to the logician's truisms, we lost control and found ourselves ascribing to actions and happenings properties which can belong only to the stock in trade of logicians, namely statements or propositions. We were talking in the logician's tone of voice about

what makes things happen, and then in the nursemaid's tone of voice about connexions between truths. Similarly here we have been talking, so to speak, in one breath with the sporting reporter of a newspaper, and in another breath with our mathematics master, and so find ourselves describing a sprint in terms of numerators and denominators and of relations between fractions in terms of efforts and despairs.